

Temporal-varying failures of nodes in networks

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We consider networks in which random walkers are removed because of the failure of specific nodes. We interpret the rate of loss as a measure of the importance of nodes, a notion we denote as *failure-centrality*. We show that the degree of the node is not sufficient to determine this measure and that, in a first approximation, the shortest loops through the node have to be taken into account. We propose approximations of the failure-centrality which are valid for temporal-varying failures and we dwell on the possibility of externally changing the relative importance of nodes in a given network, by exploiting the interference between the loops of a node and the cycles of the temporal pattern of failures. In the limit of long failure cycles we show analytically that the escape in a node is larger than the one estimated from a stochastic failure with the same failure probability. We test our general formalism in two real-world networks (air-transportation and e-mail users) and show how communities lead to deviations from predictions for failures in hubs.

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I. INTRODUCTION

Random walks in networks are the core of many complex-systems models[1]. Here we consider the case of *open* networks in which there is a positive probability of a random-walker being lost (or equivalently to freeze its movement) on a node (e.g., because of its *failure*). This situation is also known as the *trapping problem*, which after the pioneering work of Montroll [2], has received much recent attention [3, 4] [34]. Its relevance is apparent, for instance, in communication networks where a failure could correspond to the loss of an information package. Despite the negative connotation of the term *failure*, the same picture describes also situations in which the removal of random walkers is deliberately done. For instance, the removal of the information package at a node could reflect the successful arrival at the target, similar to search-related problems [5–7]. These ideas are not restricted to communication networks, e.g. in epidemic modeling failure of specific nodes could correspond to infected locations or the loss of a virus due to immunisation.

Measures for the importance (or centrality) of nodes and edges are also often defined based on random walks [1, 8–10]. In many situations the importance of a node or link is manifest only after its *failure*. This motivates us to consider the rate of the exponential loss from a network in which a node fails (escape rate) as a measure of the *failure-centrality* of this node. The traditional analysis of trapping problems is based on the first passage time to the considered node [1, 11]. A quan-

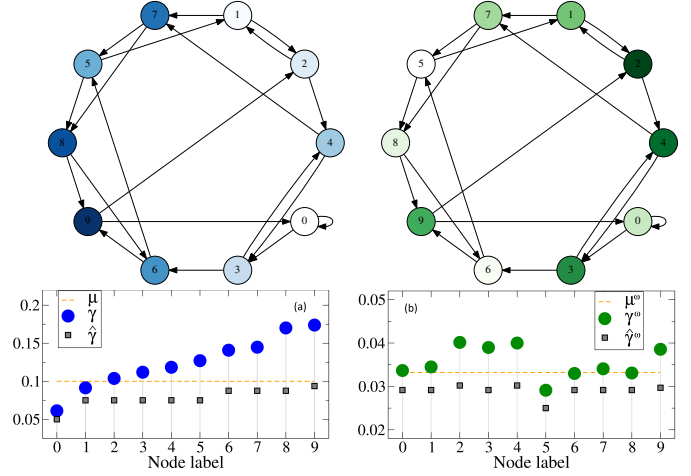


FIG. 1: (Colour online) *Node dependency of the escape rate γ* . The ten nodes of a directed regular network with (in- and out-) degree 2 are labeled according to the ranking of the escape rate in the case of complete failure. Panels (a,b) full circles show γ obtained when each node fails (a) completely or (b) periodically in time, with period three ($\omega = [110]^\infty$). Zero-order (dashed line) and first-order (squares) approximations are shown (see main text). For such small networks, the first order approximation of the *value* is not substantially better, but the *ranking* of nodes agrees much better with the observations. Nodes are colored according to γ .

tity closely related to the escape rate considered here is the global mean first-passage time, and a typical problem is how this quantity scales with network size (see e.g. Refs. [3, 4]). Mean-field models allow for an estimation of the mean first passage time (and also the escape rate) for nodes of a given degree (see, e.g., Refs. [12, 13]). A point that seems to have not been explored in detail so far is that nodes with the same degree may show very

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different behavior, i.e., our failure-centrality differs from the degree of the node. An example of this variability is shown in Fig. 1. Nodes leading to a faster decay of the number of surviving walkers are the most important one, either because they lead to the strongest leakage of the network or because they provide the fastest target for random-walkers.

The situation sketched above, equivalent to an absorbing Markov chain, has a parallel in the problem of placing holes in the phase space of chaotic dynamical systems, as previously noticed and explored in Refs. [14–17]. In that context, the dependence of the asymptotic escape rate on the size and position of the hole has been a topic of much recent interest [15, 18–22]. One insightful question is [18]: Where to place a hole to achieve a maximal escape rate? In the network context, this corresponds to asking [14]: Which is the best node to place the target on so that it is faster/slower for a random walker to find it? Some analogies between open dynamical systems and networks has been used also in applications of isospectral transformations that reduce the complexity of networks [16] and in analyses of the connectivity of the network [17].

In this paper we investigate the network properties responsible for the variety in the escape rate obtained when nodes fail. We adapt previously known results to the network problem (as in Refs. [14–17]) and we then extend them to temporal-varying failures. In particular, we show how the escape rate obtained when one node fails can be approximated based on the short loops passing through this node, one element which can often be neglected in random networks but which is present in virtually all real world examples [1, 23]. We dwell on the possibility of externally changing the relative importance of nodes in a given network, by exploiting the interference between the loops traversing the nodes and externally-imposed failures. Finally, we test our results in two real-world networks (air transportation and e-mail) and show how community structures influence our estimations.

II. SETTING

The process we study is a random walk on the network defined by an $N \times N$ irreducible and aperiodic stochastic matrix \mathbf{A}_1 : its entries a_{jk} are the probability for a random walker to jump from node j to node k at every (discrete) time step. Starting from a weighted, fully-connected, directed network with N nodes – defined by the $N \times N$ connectivity matrix \mathbf{W} with real-valued entries w_{jk} – \mathbf{A}_1 may be constructed considering w_{jk} to be proportional to the probability to follow the outgoing link ($a_{jk} = w_{jk} / \sum_k w_{jk}$). We call the dynamics in \mathbf{A}_1 the *closed* dynamics. We can *open* it by letting a node *fail*: the failure of a node j is modeled by modifying its outgoing links a_{jk} – eventually as a function of time – in such a way that $\sum_k a_{jk} < 1$. This corresponds to the physical intuition that a walker on the failed node has a non-zero probability to be lost from the network.

In a network with a failed node j , the probability $P_n(j)$ that a random walker is not lost up to time n asymptotically decays exponentially with a rate

$$\gamma(j) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln(P_n(j)). \quad (1)$$

We interpret the escape rate $\gamma(j)$ as the *failure-centrality* of the node j , i.e., a measure capturing the sensitivity of the network to its failure.

Consider first the case of total failure of a node j , corresponding to the case where all its outgoing links have zero weight (equivalent to certain loss of the random walker once on the node); and denote by $\mathbf{A}_0(j)$ the corresponding matrix. It is well known that

$$\gamma(j) = - \ln(\|\mathbf{A}_0(j)\|) \quad (2)$$

where $\|\cdot\|$ denotes the largest eigenvalue of the matrix. Note that, according to the Perron-Frobenius Theorem, $\|\mathbf{A}_1\| = 1$ corresponding to no loss of walkers moving randomly through the network. Generically $\gamma(j)$ is a non trivial function of the node, even for very simple, regular networks such as that displayed in Fig. 1a. In applications, it is important to determine which node is more relevant (in the above sense) for a given network, and more generally their ranking in relative importance. In principle, this requires generating the open network $\mathbf{A}_0(j)$ and computing Eq. (2) for $j = 1 \dots N$ nodes. In practice, for large N this requires a prohibitive computational effort (see [19] for an analytic approach that works in some particular cases). Furthermore, such brute-force computation brings no insight on the properties of the closed network \mathbf{A}_1 responsible for the importance of a node. Here we show how the values and ordering of $\gamma(j)$ can be efficiently estimated based on the degree and short loops of node j in \mathbf{A}_1 .

III. ESCAPE RATE APPROXIMANTS

As a first approximation we can imagine that, if the network is complex enough, at each time step it looses a fraction of walkers corresponding to the equilibrium measure $\mu(j)$ of the closed network on the failed node j , so that asymptotically in time $P_n(j) \sim (1 - \mu(j))^n$ and thus for a large network ($\mu(j) \ll 1$) we obtain as the zero-th order approximation

$$\gamma(j) \simeq \mu(j). \quad (3)$$

In networks with fixed degree (k -regular networks), Eq. (3) predicts the same γ for all nodes j , as does eigenvector centrality and PageRank of the closed network [1]. In chaotic systems, equivalent approximations have been shown to provide only a rough estimate of $\gamma(j)$ which, even in the case of uniform stationary distribution, show interesting fluctuations (see Ref. [20] for references and an historical overview, and Refs. [18, 24], for recent rigorous connections between the fluctuations and *periodic*

orbits). Adapting these results to the problem of random-walk in complex networks, we obtain that a better approximation $\hat{\gamma}$ is

$$\hat{\gamma}(j) = \mu(j) (1 - (\mathbf{A}^{n_j})_{jj}) \quad (4)$$

where n_j is the length of the shortest loop through the node j and $(\mathbf{A}^{n_j})_{jj}$ is the probability for a walker to return to the node j after n_j steps, through its shortest loops. An example of the approximation (4) is illustrated in Fig. 1a together with the zero- and first-order approximations.

In the following, we consider the case of temporal dependent failures. In chaotic dynamical systems, holes having a deterministic [22] and stochastic [21] variation in the *position* have been considered. Here we fix the failing node and consider a (stochastic and deterministic) temporal variation in its state (failing or not).

A. Temporal-varying failures - fixed protocol

Consider the situation in which a given temporal fixed protocol of failures, defined by a periodic binary sequence $\{\omega_t\} := \omega_1\omega_2\cdots\omega_p\omega_1\omega_2\cdots\omega_p\cdots = [\omega_1\cdots\omega_p]^\infty$ is externally imposed on a given node j . More precisely, at each time t a random walker on the network moves in the closed network \mathbf{A}_1 if $\omega_t = 1$ or in the open $\mathbf{A}_0(j)$ otherwise. In such a situation, the probability for the walker not to be lost after n iterations decays exponentially with a rate $\gamma^\omega(j)$ derived from the corresponding product of the matrices

$$\gamma^\omega(j) = -\frac{1}{p} \ln \left(\left\| \prod_{t=1}^p \mathbf{A}_{\omega_t} \right\| \right), \quad (5)$$

where, to lighten notation, we avoid explicitly expressing the dependence on the node and simply denote A_0 the open network.

We now derive approximations of Eq. (5) in the same spirit as the approximations to Eq. (2) described above. The zero-th order approximation analogous to Eq. (3) is obtained re-scaling the equilibrium measure by the fraction of time $r_\omega := \sum_{t=1}^p (1 - \omega_t)/p$ the node j fails over a period p as

$$\gamma(j) \simeq \mu(j)/r_\omega. \quad (6)$$

A correction term should take into account loops traversing the failed node, as in Eq. (4). Note however that in a temporal varying situation it is possible that, for a given protocol ω , starting with a failure event, the node j is not failing again after a complete traversal of its shortest loop. Indeed, let L_j be the *set* of lengths of all primitive loops (i.e. that are not a repetition of smaller loops) of the node j . For each time $t \leq p$ such that $\omega_t = 0$ in a given protocol ω , consider the minimum time s such that $s = kl$ for some $k \in \mathbb{N}^+$ and $l \in L_j$ and such that

$\omega_{t+s} = 0$. This time s corresponds to the length of a loop connecting the node j to itself in times in which it fails and is thus a generalization to the shortest loop in Eq. (4). We denote by $T^\omega(j)$ the set of all such times s for the node j under the failing protocol ω . Note that the cardinality of T , $|T|$, is equal to the number of failures in a period of ω . For example, consider a node with $L = \{2, 5\}$. For a protocol $\omega = [011]^\infty$ then $T = \{6\}$, while for $\omega = [01011]^\infty$ then $T = \{2, 5\}$. Finally, for every $s \in T^\omega(j)$, $(\mathbf{A}^s)_{jj}$ denotes the probability for the walker to return to the node j in s steps. An heuristic formula approximating $\gamma^\omega(j)$ considers the average over such probabilities as follows:

$$\hat{\gamma}^\omega(j) = \frac{\mu(j)}{r_\omega} \left(1 - \frac{1}{|T|} \sum_{s \in T} (\mathbf{A}^s)_{jj} \right). \quad (7)$$

The case of trivial protocol $\omega = [0]^\infty$, $|T| = 1$, s equals the shortest loop, and Eqs. (7) and (4) coincide. For non-trivial protocols, the correction term in Eq.(7) accounts for the loops in the network that traverse the blinking node j at times corresponding to failure events.

We tested Eq.(7) in two standard ensembles of random networks: undirected with power-law degree distribution (scale free); and directed k-regular (regular), i.e., all nodes with the same in and out-degree). Firstly, we assess the quality of Eq. (7) – which is based only on local properties – as an approximation of the global escape rate. The percentage error of the naive approximation based on μ – Eqs. (3) and (6) – and of our approximation are shown in Table I. Scale-free networks show a large inaccuracy in the naive approximations, which is greatly improved with our proposed formula. For the regular networks, the naive approximation already provides a quite accurate approximation, yet Eq.(7) still represents an improvement. Both approximations overestimate γ in the scale free network and underestimate it in the regular networks (see Fig. 1). The important factor determining this difference is not the degree distribution, but the directionality of the links (present in the regular but not in the scale free case). Random walkers can travel back and forth through undirected links and therefore undirected networks have a higher number of short loops (e.g., every link creates a period 2 loop), which tend to reduce γ [as in Eq. (4)]. The underestimation observed in the directed regular network agrees with what is known for the complete failure case in chaotic systems [15]).

B. Switching the relative importance of nodes

The relative importance of a node is not an intrinsic property of the network as it can depend upon the type of node failure. The most interesting consequence of this is that the rank order of two nodes may be switched if the failure becomes time dependent (compare panels (a) and (b) of Fig. 1). To elucidate the basic mechanism underlying such switching phenomena, consider the simple case

Error Between →	(γ, μ)	$(\gamma, \hat{\gamma})$	$(\gamma^\omega, \mu^\omega)$	$(\gamma^\omega, \hat{\gamma}^\omega)$
Network Topology ↓	(complete failure)		(temporal failure)	
Scale free	33.6(4)	5.5(1)	13.0(1)	1.24(3)
Regular	0.695(6)	0.650(1)	0.336(2)	0.3301(4)

TABLE I: (Colour online) *Approximation accuracy.* Percentage error between the approximations $(\mu, \hat{\gamma})$ and true (γ) escape rate computed as an average over all nodes in the network. Each node is considered to fail at all times (complete failure) or periodically in time with protocol $\omega = [01011]^\infty$ (temporal failure). Random networks with 1,000 nodes and two different degree distributions $P(\deg(i) = k)$ were used: scale-free networks are undirected and have $P(\deg(i) = k) \sim k^{-2.5}$ while regular networks have out-degree equal to in-degree equal to 3. The reported values are the average and the standard error of the mean (in brackets, corresponding to an uncertainty in the last digit) computed over an ensemble of networks (94 generated according to Ref. [25] for scale free and 100 generated according to Ref. [26] for regular networks.)

in which all nodes have the same equilibrium measure μ , as in the k -regular network discussed above. Suppose for definiteness that two nodes i, j have rates $\hat{\gamma}(i) > \hat{\gamma}(j)$ and shortest loops $n(i) < n(j)$, as expected from Eq. (4). In this situation, from Eq. (7) we see that a periodic protocol of period $p = n(j)$ and with a single failure over the period p , will induce a switching on the relative importance of the node, that is $\hat{\gamma}^\omega(j) > \hat{\gamma}^\omega(i)$. Here we are using the fact that the importance of a node depends on both its invariant measure and first return-time probability, see Eq. (4). As an example, Fig. 1(b) shows how the importance of nodes 2 ($n_2 = 2$) and 7 ($n_7 = 3$) are switched by the protocol $\omega = [110]^\infty$. By exploiting the interference between the loops of a node and the periods of the temporal pattern of failures it is thus possible to devise protocols ω that change the relative importance of nodes in a given network.

We now quantify the extent to which Eq. (7) describes switches. For each node j of each of the two networks discussed in Tab. I we compute $\gamma^\omega(j)$ and $\hat{\gamma}^\omega(j)$ for the protocol $\omega = [01011]^\infty$ and $\gamma(j)$ for the fully open case (i.e., protocol $[0]^\infty$). This led to three different rankings of the nodes for each network. We first confirm that the ranking of nodes estimated from $\hat{\gamma}^\omega$ is very close to the true one obtained from γ^ω . This is quantified by the Kendall rank correlation τ [35] between these two rankings, that yields $\tau = 0.99920(5)$ (scale free network) and $\tau = 0.926(1)$ (regular network), where the distance to $\tau = 1$ is proportional to the number of switches needed to map the estimated into the true ranking. Next we investigate the magnitude of switchings introduced by the time-periodic protocol ω and the extent to which our estimation (7) is able to describe them. We obtain that the protocol leads to a substantial change in the ranking in the regular network and small but still detectable change in the scale free network. This is quantified by the *Spearman's* rank correlation coefficient $0 \leq \rho \leq 1$ [36] between the ranking obtained in the fully (γ) and temporal (γ^ω) networks:

$\rho = 0.9640(5)$ (scale free) and $\rho = 0.548(3)$ (regular), where $\rho = 1$ corresponds to perfect agreement between the rankings. The ranking obtained from $\hat{\gamma}^\omega$ in Eq. (7) yields a good estimation of these values: $\rho = 0.9501(8)$ (scale free) and $\rho = 0.531(3)$ (regular) (ρ computed between the ranks obtained from $\hat{\gamma}^\omega$ and γ).

C. Temporal-varying failure - stochastic

The limit of large periods p in the periodic-varying scenario studied above can be thought of as a typical realization of a random failure. Considering that the limit $p \rightarrow \infty$ is taken preserving (on average) the fraction of times q the node fails, the corresponding stochastic process is that of a node that *blinks* on and off at each time step randomly, such that at each time it is open with probability q . Let \mathbf{G}_n be an n -fold product of \mathbf{A}_0 and \mathbf{A}_1 where the matrices are chosen randomly with probability $(q-1)$ and q respectively, and let $\langle \cdot \rangle$ be the average over such products. The rate $\gamma^R(j)$ is then defined as the average rate taken over all possible \mathbf{G}_n in the limit as n goes to infinity and is given by

$$\gamma^R(j) = \lim_{n \rightarrow \infty} -\frac{\langle \ln \|\mathbf{G}_n\| \rangle}{n}. \quad (8)$$

A picture similar to such a (quenched) case is the (annealed) one in which at every time walkers on the failing node are lost with probability q . Equivalently, this corresponds to reducing by a factor $1 - q$ the probabilities to leave the failed node j , that is $a_{jk} \rightarrow (1 - q)a_{jk}$ for all k . This situation can be thought of as the failing node having a degree q of *permeability*. In this case the rate is

$$\gamma^P(j) = -\ln(\|(1 - q)\mathbf{A}_1 + q\mathbf{A}_0(j)\|). \quad (9)$$

The stochasticity in the two scenarios leading to Eqs. (8) and (9) differ and satisfy a simple inequality [37]

$$\gamma^R(j) \geq \gamma^P(j) \quad \forall j, \quad (10)$$

see Ref. [27] for an equivalent result in chaotic systems and Refs. [28, 29] for results on products of random matrices. From the point of view of periodic protocols, this implies that for long periods p the escape rate of most nodes will be larger than the one from the averaged operator.

IV. REAL-WORLD NETWORKS

We now test our theory in two networks with topologies created by real interactions. Firstly we study the transport network of the 500 busiest commercial airports in the United States in the year 2002 [30][38]. Airports (nodes) are connected (edges) with a strength $w_{i,j} = w_{j,i}$ proportional to the (yearly) number of existing seats in flights between them. The failure of a node corresponds

to an airport to which aeroplanes arrive but are unable to leave. We then study the communication network of e-mail users at the University Rovira i Virgili [31][39]. The nodes here represent individual e-mail users. An undirected edge exists between users who have exchanged at least one e-mail. Our main motivation for studying these networks is to test our theory in real-world networks with non-trivial topologies.

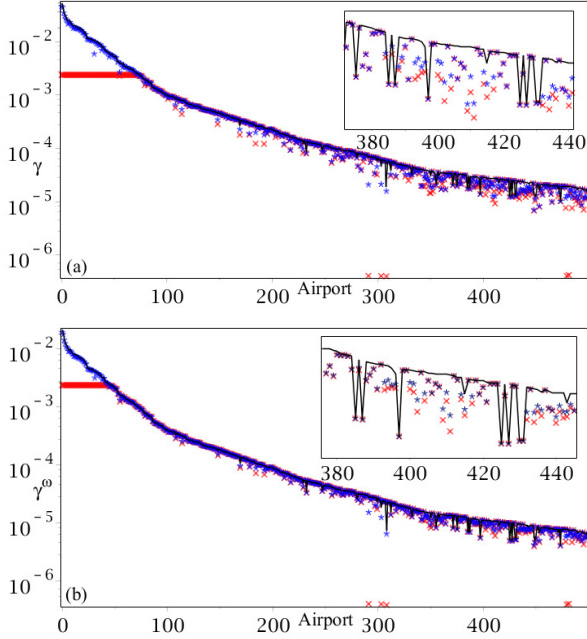


FIG. 2: (Colour online). Escape rate in the air-transportation network. The plots show the escape rate γ^ω (red crosses), the equilibrium measure μ as in Eq. (4) (black line), and the approximation $\hat{\gamma}$ from Eq. (7) (blue asterisk) obtained if each of the airports fails. (a) Fully open network $\omega = [0]^\infty$. (b) Temporal-varying failures with protocol $\omega = [01011]^\infty$. The error – as in Tab. I – between (γ, μ) is 160 and between $(\gamma, \hat{\gamma})$ is 98.6. The *Spearman's* rank correlation coefficient $\rho_{A,B}$ between the ranks obtained from measures A and B are: $\rho_{\text{degree}, \gamma^\omega} = 0.765$, $\rho_{\mu, \gamma^\omega} = 0.984$, and $\rho_{\hat{\gamma}, \gamma^\omega} = 0.997$ for $\omega = [0]^\infty$; and $\rho_{\text{degree}, \gamma^\omega} = 0.781$, $\rho_{\mu, \gamma^\omega} = 0.993$, and $\rho_{\hat{\gamma}, \gamma^\omega} = 0.997$ for $\omega = [01011]^\infty$. Finally, comparing γ^ω of the two protocols we obtain $\rho = 0.997$. Values reported to 3 s.f.

The results shown in Fig. 2 and Fig. 3 confirm our conclusions obtained in the random networks for failures in almost all nodes. In particular, we obtain that the approximation $\hat{\gamma}$ in Eq. (7) is better than μ in Eq. (4) and that (temporal) failures induce switches in the rank of the nodes. As we saw in the scale-free networks previously, both approximations μ and $\hat{\gamma}$ systematically overestimate γ .

On the other hand, a new observation in the air-transportation network is the existence of sets of nodes which deviate substantially from the approximations (the outliers in Fig. 2): the airports from 1 to 73 (hubs) and the set S of five airports $S = \{291, 303, 308, 479, 481\}$.

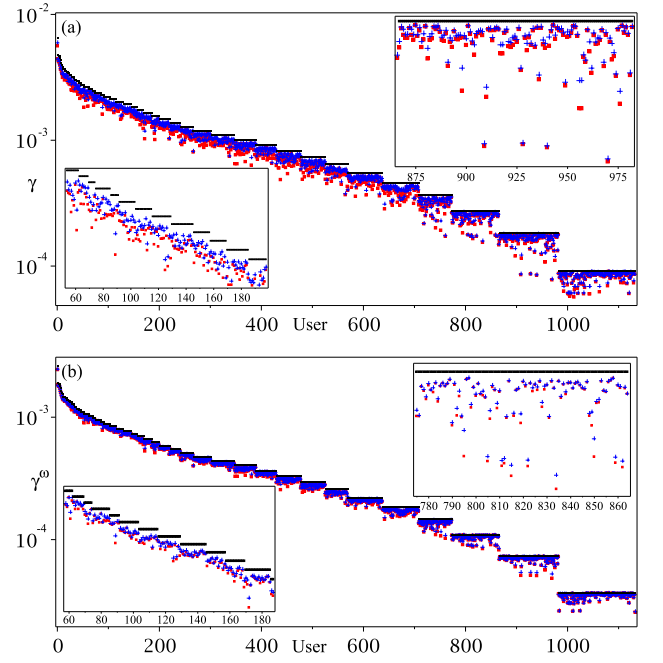


FIG. 3: (Colour online). Escape rate in the e-mail communication network. Illustrated here is the escape rate γ^ω (red squares), the equilibrium measure μ as in Eq. (4) (black line), and the approximation $\hat{\gamma}$ from Eq. (7) (blue crosses) obtained if an individual e-mail user fails. (a) Fully open network $\omega = [0]^\infty$. (b) Temporal-varying failures with protocol $\omega = [01011]^\infty$. The error – as in Tab. I – between (γ, μ) is 18.7 and between $(\gamma, \hat{\gamma})$ is 4.52. The *Spearman's* rank correlation coefficient $\rho_{A,B}$ between the ranks obtained from measures A and B are: $\rho_{\text{degree}, \gamma^\omega} = 0.997$, $\rho_{\mu, \gamma^\omega} = 0.997$, and $\rho_{\hat{\gamma}, \gamma^\omega} = 0.999$ for $\omega = [0]^\infty$; and $\rho_{\text{degree}, \gamma^\omega} = 0.781$, $\rho_{\mu, \gamma^\omega} = 0.993$, and $\rho_{\hat{\gamma}, \gamma^\omega} = 1.000$ for $\omega = [01011]^\infty$. Finally, comparing γ^ω of the two protocols we obtain $\rho = 0.999$. Values reported to 3 s.f.

Next we explain these deviations and their relation with the existence of communities in the network, i.e., nodes that are strongly connected to each other and only weakly connected to other nodes[1]. In the air transportation network the nodes S are the most interconnected ones, having a single weak connection (through node 13) to the rest of the network (a bottleneck). The escape rate from S to the outside is $\gamma^* = 0.00238$ (obtained, e.g., if we start in S and consider the node 13 to fail). If a node outside S fails, two cases have to be considered. For nodes with small degree, the escape rate from the large portion of the network is $\gamma < \gamma^*$ and therefore it dominates the global escape rate and agrees with the approximations. For nodes with high degree, the escape from the large portion of the network is fast, $\gamma > \gamma^*$, and hence the global rate becomes $\gamma = \gamma^*$ because asymptotically the smallest escape rate dominates. This explains the appearance of the plateau in the hubs of Fig. 2 at $\gamma \approx \gamma^*$. In this case the approximations μ and $\hat{\gamma}$ describe

closely the escape rate obtained from the *second* largest eigenvalue of the network whose eigenvector is localized outside S (the eigenvector associated to γ^* is always localized in S). Finally, if a node in S fails, the escape rate is dominated by the small probability to jump from the large network through the bottleneck (in node 13) to S , explaining the appearance of the same small γ for all these nodes (bottom crosses in Fig. 2ab).

In order to understand further the effect that the topology of these real-world networks has, we repeat our analysis in link randomised versions of the airport and e-mail networks. In both cases we obtain that the escape rate is very well approximated by the zeroth order approximation (3) (Speraman's rank correlation of 1.000 in the randomized airport network and 0.997 in the e-mail network). This happens because the randomization destroys communities and short loops in the network, the sources of deviations from the zero-order approximation discussed above. Short loops are ubiquitous in real-world networks, in which improved results are obtained by our approximation (7).

V. CONCLUSIONS

In summary, we have shown how to efficiently estimate the escape rate of random walkers in networks in which specific nodes fail. Our new formula (7) outperforms estimations based on the degree or equilibrium measures, shows the importance of the degree and of short loops passing through the failing nodes, and applies also to temporal dependent failures (stochastic or through a fixed protocol). The approximation is valid for small perturbations (i.e., degree of the failing node much smaller than network size). In this limit, our re-

sults are expected to hold also in cases in which multiple nodes and edges fail.

The escape rate can be viewed as a measure of node importance that quantifies how sensitive the network is to failures in this node (*failure-centrality*). The results summarized above allow for an estimation of this measure, which differs from the degree and other centrality measures. For instance, the eigenvector and page-rank centrality coincide with the zero-th order approximation of the failure centrality, Eq.(3), and are therefore different from the failure centrality already in this simple case. In networks with general degree distributions these measures differ even more significantly from the failure centrality, e.g., in the scale-free networks discussed in Table I we found a Spearman's rank correlation of 0.735 (to page rank) and 0.748 (to eigenvector centrality). The failure-centrality measure depends also on the type of temporal failure the nodes experience. We have shown how switches in the importance of nodes appear and can be engineered depending on the interplay between the period of the temporal failure and the length of the loops present in the network.

In real-world networks, the appearance of communities enhance the deviations between the escape rate and naive estimations. At the same time, when hubs fail the escape of random walks from the closest communities may dominate the global escape rate and lead to deviations from our theory. The introduction of failures in nodes can thus be used to quickly detect the strongest communities of a network, in the same way that holes have been introduced in chaotic dynamical systems to visualize their underlying invariant manifolds [20].

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- [34] This is not to be confused with the process of key-removal, where nodes and edges are removed but the network remains closed and no trapping mechanism is in action (see for example [32], [33])
- [35] The *Kendall's τ* rank correlation coefficient between two ranked data sets $(x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n)$ is defined as $\tau = \frac{n_c - n_d}{\frac{1}{2}n(n-1)}$, where n_c is the number of concordant pairs and n_d the number of discordant pairs. (The total number of pairs is $n(n-1)/2 = n_d + n_c$). A pair of elements (i, j) is concordant if their order in both ranking agree (i.e. $x_i \leq x_j \Rightarrow y_i \leq y_j$), otherwise it is discordant.
- [36] The *Spearman's ρ* rank correlation coefficient between two ranked data sets of size n is defined as $\rho = 1 - \frac{6 \sum_i d_i^2}{n(n^2-1)}$, where d_i is the difference in the ranking between the i^{th} variables.
- [37] To see this, notice that: (i) the term inside the logarithmic function in Eq. (9) is for $n \rightarrow \infty$ equal to $\|((1-q)\mathbf{A}_1 + q\mathbf{A}_0)^n\|^{\frac{1}{n}} = \langle \|\mathbf{G}_n\| \rangle^{\frac{1}{n}}$; and (ii) the negative logarithm is a convex function and thus (from Jensen inequality) $-\ln \langle \|\mathbf{G}_n\| \rangle \geq \langle -\ln \|\mathbf{G}_n\| \rangle$.
- [38] The data and further details are accessible at (retrieved Nov. 2014) <https://sites.google.com/site/cxnets/usairtransportationnetwork>
- [39] The data and further details are accessible at (retrieved May. 2015) <http://konect.uni-koblenz.de/networks/arenas-email>